

# On the counting function of the embedded eigenvalues for some manifold with cusps, and magnetic Laplacian

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## Abstract

We consider a non compact, complete manifold  $\mathbf{M}$  of finite area with cuspidal ends. The generic cusp is isomorphic to  $\mathbf{X} \times ]1, +\infty[$  with metric  $ds^2 = (h + dy^2)/y^{2\delta}$ .  $\mathbf{X}$  is a compact manifold with nonzero first Betti number equipped with the metric  $h$ . For a one-form  $A$  on  $\mathbf{M}$  such that in each cusp  $A$  is a non exact one-form on the boundary at infinity, we prove that the magnetic Laplacian  $-\Delta_A = (id + A)^*(id + A)$  satisfies the Weyl asymptotic formula with sharp remainder. We deduce an upper bound for the counting function of the embedded eigenvalues of the Laplace-Beltrami operator  $-\Delta = -\Delta_0$ .<sup>1</sup>

## 1 Introduction

We consider a smooth, connected  $n$ -dimensional Riemannian manifold  $(\mathbf{M}, \mathbf{g})$ , ( $n \geq 2$ ), such that

$$\mathbf{M} = \bigcup_{j=0}^J \mathbf{M}_j \quad (J \geq 1), \quad (1.1)$$

where the  $\mathbf{M}_j$  are open sets of  $\mathbf{M}$ . We assume that the closure of  $\mathbf{M}_0$  is compact and that the other  $\mathbf{M}_j$  are cuspidal ends of  $\mathbf{M}$ .

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This means that  $\mathbf{M}_j \cap \mathbf{M}_k = \emptyset$ , if  $1 \leq j < k$ , and that there exists, for any  $j$ ,  $1 \leq j \leq J$ , a closed compact  $(n-1)$ -dimensional Riemannian manifold  $(\mathbf{X}_j, \mathbf{h}_j)$  such that  $\mathbf{M}_j$  is isometric to  $\mathbf{X}_j \times ]a_j^2, +\infty[$ , ( $a_j > 0$ ) equipped with the metric

$$ds_j^2 = y^{-2\delta_j}(\mathbf{h}_j + dy^2) ; \quad (1/n < \delta_j \leq 1). \quad (1.2)$$

So there exists a smooth real one-form  $A_j \in T^*(\mathbf{X}_j)$ , non exact, such that

$$\begin{cases} i) dA_j \neq 0 \\ \text{or} \\ ii) dA_j = 0 \text{ and } [A_j] \text{ is not integer.} \end{cases} \quad (1.3)$$

In *ii*) we mean that there exists a smooth closed curve  $\gamma$  in  $\mathbf{X}_j$  such that

$$\int_{\gamma} A_j \notin 2\pi\mathbb{Z}.$$

Then one can always find a smooth real one-form  $A \in T^*(\mathbf{M})$  such that

$$\forall j, 1 \leq j \leq J, \quad A = A_j \quad \text{on } \mathbf{M}_j. \quad (1.4)$$

We define the magnetic Laplacian, the Bochner Laplacian

$$-\Delta_A = (i d + A)^*(i d + A), \quad (1.5)$$

( $i = \sqrt{-1}$ ,  $(i d + A)u = i du + uA$ ,  $\forall u \in C_0^\infty(\mathbf{M}; \mathbb{C})$ , the upper star,  $*$ , stands for the adjoint in  $L^2(\mathbf{M})$ ).

As  $\mathbf{M}$  is a complete metric space, by Hopf-Rinow theorem  $\mathbf{M}$  is geodesically complete, so it is well known, (see [Shu]), that  $-\Delta_A$  has a unique self-adjoint extension on  $L^2(\mathbf{M})$ , containing in its domain  $C_0^\infty(\mathbf{M}; \mathbb{C})$ , the space of smooth and compactly supported functions. The spectrum of  $-\Delta_A$  is gauge invariant : for any  $f \in C^1(\mathbf{M}; \mathbb{R})$ ,  $-\Delta_A$  and  $-\Delta_{A+df}$  are unitary equivalent, hence they have the same spectrum.

For a self-adjoint operator  $P$  on a Hilbert space  $H$ ,  $\text{sp}(P)$ ,  $\text{sp}_{\text{ess}}(P)$ ,  $\text{sp}_p(P)$  and  $\text{sp}_d(P)$  will denote respectively the spectrum, the essential spectrum, the point spectrum and the discrete spectrum of  $P$ . We recall that  $\text{sp}(P) = \text{sp}_{\text{ess}}(P) \cup \text{sp}_d(P)$ ,  $\text{sp}_d(P) \subset \text{sp}_p(P)$  and  $\text{sp}_{\text{ess}}(P) \cap \text{sp}_d(P) = \emptyset$ .

**Theorem 1.1** *Under the above assumptions on  $\mathbf{M}$ , the essential spectrum of the Laplace-Beltrami operator on  $\mathbf{M}$ ,  $-\Delta = -\Delta_0$  is given by*

$$\begin{cases} \text{sp}_{\text{ess}}(-\Delta) = [0, +\infty[, & \text{if } 1/n < \delta < 1 \\ \text{sp}_{\text{ess}}(-\Delta) = [\frac{(n-1)^2}{4}, +\infty[, & \text{if } \delta = 1 \end{cases}. \quad (1.6)$$

When (1.3) and (1.4) are satisfied, the magnetic Laplacian  $-\Delta_A$  has a compact resolvent. The spectrum  $\text{sp}(-\Delta_A) = \text{sp}_d(-\Delta_A)$  is a sequence of non-decreasing eigenvalues  $(\lambda_j)_{j \in \mathbb{N}^*}$ ,  $\lambda_j \leq \lambda_{j+1}$ ,  $\lim_{j \rightarrow +\infty} \lambda_j = +\infty$ , such that the sequence of normalized eigenfunctions  $(\varphi_j)_{j \in \mathbb{N}^*}$  is a Hilbert basis of  $L^2(\mathbf{M})$ . Moreover  $\lambda_0 > 0$ .

For any self-adjoint operator  $P$  with compact resolvent, and for any real  $\lambda$ ,  $N(\lambda, P)$  will denote the number of eigenvalues, (repeated according to their multiplicity), of  $P$  less than  $\lambda$ ,

$$N(\lambda, P) = \text{trace}(\chi_{]-\infty, \lambda[}(P)) , \quad (1.7)$$

(for any  $I \subset \mathbb{R}$ ,  $\chi_I(x) = 1$  if  $x \in I$  and  $\chi_I(x) = 0$  if  $x \in \mathbb{R} \setminus I$ ).

The asymptotic behavior of  $N(\lambda, -\Delta_A)$  satisfies the Weyl formula with the following sharp remainder.

**Theorem 1.2** *Under the above assumptions on  $\mathbf{M}$  and on  $A$ , we have the Weyl formula with remainder as  $\lambda \rightarrow +\infty$ ,*

$$N(\lambda, -\Delta_A) = |\mathbf{M}| \frac{\omega_n}{(2\pi)^n} \lambda^{n/2} + \mathbf{O}(\mathbf{r}(\lambda)) , \quad (1.8)$$

with

$$r(\lambda) = \begin{cases} \lambda^{(n-1)/2} \ln(\lambda), & \text{if } 1/(n-1) \leq \delta \\ \lambda^{1/(2\delta)}, & \text{if } 1/n < \delta < 1/(n-1) \end{cases} , \quad (1.9)$$

$\delta = \min_{1 \leq j \leq J} \delta_j$ ,  $|\mathbf{M}|$  is the Riemannian measure of  $\mathbf{M}$  and  $\omega_d$  is the euclidian

volume of the unit ball of  $\mathbb{R}^d$ ,  $\omega_d = \frac{\pi^{d/2}}{\Gamma(1 + \frac{d}{2})}$ .

The asymptotic formula (1.8) without remainder is given in [Go-Mo], and with remainder but only for  $n = 2$  (and  $\delta_j = 1$  for any  $1 \leq j \leq J$ ) in [Mo-Tr].

The Laplace-Beltrami operator  $-\Delta = -\Delta_0$  may have embedded eigenvalues in its essential spectrum  $\text{sp}_{\text{ess}}(-\Delta)$ . Let  $N_{\text{ess}}(\lambda, -\Delta)$  denote the number of eigenvalues of  $-\Delta$ , (counted according to their multiplicity), less than  $\lambda$ .

**Theorem 1.3** *There exists a constant  $C_{\mathbf{M}}$  such that, for any  $\lambda \gg 1$ ,*

$$N_{\text{ess}}(\lambda, -\Delta) \leq |\mathbf{M}| \frac{\omega_n}{(2\pi)^n} \lambda^{n/2} + C_{\mathbf{M}} r_0(\lambda) , \quad (1.10)$$

with  $r_0(\lambda)$  defined by

$$r_0(\lambda) = \begin{cases} \lambda^{\frac{n-1}{2}} \ln(\lambda), & \text{if } 2/n \leq \delta \leq 1 \\ \lambda^{\frac{n-(n\delta-1)}{2}}, & \text{if } 1/n < \delta < 2/n \end{cases} ; \quad (1.11)$$

$\delta$  is the one defined in Theorem 1.2 .

The above upper bound proves that any eigenvalue of  $-\Delta$  has finite multiplicity.

The estimate (1.10) is sharp when  $n = 2$ . There exist hyperbolic surfaces  $\mathbf{M}$  of finite area so that

$$N_{\text{ess}}(\lambda, -\Delta) = |\mathbf{M}| \frac{\omega_2}{(2\pi)^2} \lambda + \Gamma_{\mathbf{M}} \lambda^{1/2} \ln(\lambda) + \mathbf{O}(\lambda^{1/2}) ,$$

for some constant  $\Gamma_{\mathbf{M}}$ . See [Mul] for such examples.

Still in the case of surfaces, a compact perturbation of the metric of non compact hyperbolic surface  $\mathbf{M}$  of finite area can destroy all embedded eigenvalues, see [Coll].

## 2 Proof

### 2.1 Proof of Theorem 1.1

Since the essential spectrum of an elliptic operator on a manifold is invariant by compact perturbation of the manifold, (see for example [Do-Li], Proposition 2.1 ), we can write

$$\text{sp}_{\text{ess}}(-\Delta_A) = \bigcup_{j=1}^J \text{sp}_{\text{ess}}(-\Delta_A^{\mathbf{M}_j, D}) , \quad (2.1)$$

where  $-\Delta_A^{\mathbf{M}_j, D}$  denotes the self-adjoint operator on  $L^2(\mathbf{M}_j)$  associated to  $-\Delta_A$  with Dirichlet boundary conditions on the boundary  $\partial\mathbf{M}_j$  of  $\mathbf{M}_j$ .

Let us consider a cusp  $\mathbf{M}_j = \mathbf{X}_j \times ]a_j^2, +\infty[$  equipped with the metric (1.2). Then for any  $u \in C^2(\mathbf{M}_j)$ ,

$$-\Delta_A u = -y^{2\delta_j} \Delta_{A_j}^{\mathbf{X}_j} u - y^{n\delta_j} \partial_y (y^{(2-n)\delta_j} \partial_y u) , \quad (2.2)$$

where  $\Delta_{A_j}^{\mathbf{X}_j}$  is the magnetic Laplacian on  $\mathbf{X}_j$  : if for local coordinates  $\mathbf{h}_j =$

$\sum_{k,\ell} G_{k\ell} dx_k dx_\ell$  and  $A_j = \sum_{k=1}^{n-1} a_{j,k} dx_k$ , then

$$-\Delta_{A_j}^{\mathbf{X}_j} = \frac{1}{\sqrt{\det(G)}} \sum_{k,\ell} (i\partial_{x_k} + a_{j,k}) \left( \sqrt{\det(G)} G^{k\ell} (i\partial_{x_\ell} + a_{j,\ell}) \right).$$

We perform the change of variables  $y = e^t$ , and define the unitary operator  $U : L^2(\mathbf{X}_j \times ]2\ln(a_j), +\infty[) \rightarrow L^2(\mathbf{M}_j)$ , where  $]2\ln(a_j), +\infty[$  is equipped with the standard euclidian metric  $dt^2$ , by  $U(f) = y^{(n\delta_j-1)/2} f$ . Thus  $L^2(\mathbf{M}_j)$  is unitary equivalent to  $L^2(\mathbf{X}_j \times ]2\ln(a_j), +\infty[)$ , and

$$-U^* \Delta_A U f = -e^{2\delta_j t} \Delta_{A_j}^{\mathbf{X}_j} f + \frac{(n\delta_j - 1)[3 + \delta_j(n - 4)]}{4} e^{2t(\delta_j-1)} f - \partial_t(e^{2t(\delta_j-1)} \partial_t f). \quad (2.3)$$

Let us denote by  $(\mu_\ell(j))_{\ell \in \mathbb{N}}$  the increasing sequence of eigenvalues of  $-\Delta_{A_j}^{\mathbf{X}_j}$ , each eigenvalue repeated according to its multiplicity. Then  $-\Delta_A^{\mathbf{M}_j, D}$

is unitary equivalent to  $\bigoplus_{\ell=0}^{+\infty} L_{j,\ell}^D$ ,

$$\text{sp}(-\Delta_A^{\mathbf{M}_j, D}) = \text{sp}\left(\bigoplus_{\ell=0}^{+\infty} L_{j,\ell}^D\right), \quad (2.4)$$

where  $L_{j,\ell}^D$  is the Dirichlet operator on  $L^2(]2\ln(a_j), +\infty[)$  associated to

$$L_{j,\ell} = e^{2\delta_j t} \mu_\ell(j) + \frac{(n\delta_j - 1)}{4} [3 + \delta_j(n - 4)] e^{2t(\delta_j-1)} - \partial_t(e^{2t(\delta_j-1)} \partial_t). \quad (2.5)$$

If  $\mu_\ell(j) > 0$  then  $\text{sp}(L_{j,\ell}^D) = \text{sp}_d(L_{j,\ell}^D) = \{\mu_{\ell,k}(j); k \in \mathbb{N}\}$ , where  $(\mu_{\ell,k}(j))_{k \in \mathbb{N}}$  is the increasing sequence of eigenvalues of  $L_{j,\ell}^D$ ,  $\lim_{k \rightarrow +\infty} \mu_{\ell,k}(j) = +\infty$ .

If  $\mu_\ell(j) = 0$  then  $\text{sp}(L_{j,\ell}^D) = \text{sp}_{\text{ess}}(L_{j,\ell}^D) = [\alpha_n, +\infty[$ , with  $\alpha_n = 0$  if  $\delta_j < 1$ , and  $\alpha_n = (n-1)^2/4$  if  $\delta_j = 1$ .

Since we have  $\mu_0(j) = 0$  when  $A = 0$ , we get that  $\text{sp}_{\text{ess}}(-\Delta_0) = [\alpha_n, +\infty[$ .

If  $A$  satisfies assumptions (1.3) and (1.4), then  $0 < \mu_0(j) \leq \mu_\ell(j)$  for all  $j$  and  $\ell$ , (see for example [Hel]), so  $\text{sp}(-\Delta_A^{\mathbf{M}_j, D}) = \{\mu_{\ell,k}(j); (\ell, k) \in \mathbb{N}^2\}$ . As  $\lim_{\ell \rightarrow +\infty} \mu_{\ell,k}(j) = +\infty$ , each  $\mu_{\ell,k}(j)$  is an eigenvalue of  $-\Delta_A^{\mathbf{M}_j, D}$  of finite multiplicity, so  $\text{sp}(-\Delta_A^{\mathbf{M}_j, D}) = \text{sp}_d(-\Delta_A^{\mathbf{M}_j, D})$ . Therefore, we get that  $\text{sp}_{\text{ess}}(-\Delta_A) = \emptyset$   $\square$

## 2.2 Proof of Theorem 1.2

We proceed as in [Mo-Tr].

We begin by establishing formula (1.8) for  $\mathbf{M}_j$ , with  $-\Delta_A^{\mathbf{M}_j, D}$  defined in (2.1), instead of  $-\Delta_A$ . When  $\delta_j = 1$  we make the same change of variables and functions as in the proof of Theorem 1.1, but when  $1/n < \delta_j < 1$ , we set  $y = [(1 - \delta_j)t]^{1/(1-\delta_j)}$ , and define the unitary operator

$$U : L^2(\mathbf{X}_j \times \left[ \frac{a_j}{1 - \delta_j}, +\infty \right]) \rightarrow L^2(\mathbf{M}_j), \text{ by } U(f) = y^{(n-1)\delta_j/2} f.$$

Then when  $1/n < \delta_j < 1$ ,

$$-U^* \Delta_A U f = -[(1 - \delta_j)t]^{\frac{2\delta_j}{1-\delta_j}} \Delta_{A_j}^{\mathbf{X}_j} f + \frac{(n-1)\delta_j[(n-3)\delta_j + 2]}{4(1 - \delta_j)^2 t^2} f - \partial_t^2 f. \quad (2.6)$$

As a matter of fact,

$$-U^* y^{n\delta_j} \partial_y [y^{(2-n)\delta_j} \partial_y U(f)] = -y^{(n+1)\delta_j/2} \partial_y [y^{(3-n)\delta_j/2} \partial_y f] - \frac{(n-1)\delta_j}{2} y^{2\delta_j-1} \partial_y f + \frac{(n-1)\delta_j[(n-3)\delta_j + 2]}{4} y^{-2(1-\delta_j)} f,$$

then using that  $y^{\delta_j} \partial_y = \partial_t$  and that  $t^\rho \partial_t = \partial_t(t^\rho \cdot) - \rho t^{\rho-1}$ , we get easily (2.6).

Equality (2.4) still holds when  $L_{j,\ell}^D$  is the Dirichlet operator on  $L^2(\left[ \frac{a_j^{2(1-\delta_j)}}{1-\delta_j}, +\infty \right])$  associated to

$$L_{j,\ell} = \mu_\ell(j) [(1 - \delta_j)t]^{\frac{2\delta_j}{1-\delta_j}} + \frac{(n-1)\delta_j[(n-3)\delta_j + 2]}{4(1 - \delta_j)^2 t^2} - \partial_t^2. \quad (2.7)$$

From now on, any constant depending only on  $\delta_j$  and on  $\min_j \mu_0(j)$  will be invariably denoted by  $C$ .

As in [Mo-Tr], we will follow Titchmarsh's method. Using Theorem 7.4 in [Tit] page 146, we prove the following Lemma.

**Lemma 2.1** *There exists  $C > 1$  so that for any  $\lambda >> 1$  and any  $\ell \in K_\lambda = \{l \in \mathbb{N}; \mu_\ell(j) \in [0, \lambda / \min_j a_j^2]\}$ ,*

$$|N(\lambda, L_{j,\ell}^D) - \frac{1}{\pi} w_{j,\ell}(\lambda)| \leq C \ln(\lambda), \quad (2.8)$$

$$\text{with } w_{j,\ell}(\mu) = \int_{\alpha_j}^{+\infty} [\mu - V_{j,\ell}(t)]_+^{1/2} dt = \int_{\alpha_j}^{T_j(\mu)} [\mu - V_{j,\ell}(t)]_+^{1/2} dt.$$

The potential  $V_{j,\ell}$  is defined as following:

$$\begin{cases} \text{if } \delta_j = 1 & V_{j,\ell}(t) = \mu_\ell(j)e^{2t} + \frac{(n-1)^2}{4} \\ \text{if } 1/n < \delta_j < 1 & V_{j,\ell}(t) = \mu_\ell(j)[(1-\delta_j)t]^{\frac{2\delta_j}{1-\delta_j}} + \frac{(n-1)\delta_j[(n-3)\delta_j+2]}{4(1-\delta_j)^2}t^{-2} \end{cases}, \quad (2.9)$$

and

$$\begin{cases} \text{if } \delta_j = 1 & \alpha_j = 2\ln(a_j), \quad T_j(\mu) = \frac{1}{2}\ln(\mu/\mu_0(j)) \\ \text{if } 1/n < \delta_j < 1 & \alpha_j = \frac{a_j^{2(1-\delta_j)}}{1-\delta_j}, \quad T_j(\mu) = \frac{1}{1-\delta_j} \left( \frac{\mu}{\mu_0(j)} \right)^{\frac{1-\delta_j}{2\delta_j}} \end{cases}. \quad (2.10)$$

**Proof of Lemma 2.1**

When  $1/n < \delta_j < 1$ , by enlarging  $\mathbf{M}_0$  and reducing  $\mathbf{M}_j$ , we can take  $\alpha_j$  large enough so that  $V_{j,\ell}(t)$  is an increasing function on  $[\alpha_j, +\infty[$  and  $\lambda/\mu_\ell(j) \gg 1$  when  $\ell \in K_\lambda$ . Then, if  $\alpha_j \leq Y < X(\lambda) = V_{j,\ell}^{-1}(\lambda)$ , following the proof of Theorem 7.4 in [Tit] pages 146-147, we get that

$$|N(\lambda, L_{j,\ell}^D) - \frac{1}{\pi}w_{j,\ell}(\lambda)| \leq \quad (2.11)$$

$$C[\ln(\lambda - V_{j,\ell}(\alpha_j)) - \ln(\lambda - V_{j,\ell}(Y)) + (X(\lambda) - Y)(\lambda - V_{j,\ell}(Y)) + 1].$$

When  $\delta_j = 1$ , we choose  $Y = X(\lambda) - \frac{\sqrt{\ln \lambda}}{\sqrt{\lambda}}$ .

When  $1/n < \delta_j < 1$ , we choose  $Y = X(\lambda) - \frac{\sqrt{\ln \lambda}}{\sqrt{\lambda}} \left( \frac{\lambda}{\mu_\ell(j)} \right)^{\frac{1-\delta_j}{4\delta_j}}$ ;

$$(X(\lambda) \sim \frac{1}{1-\delta_j} \left( \frac{\lambda}{\mu_\ell(j)} \right)^{\frac{1-\delta_j}{2\delta_j}}) \square$$

Let us recall the sharp asymptotic Weyl formula of L. Hörmander [Hor1] (see also [Hor2]).

**Theorem 2.2** *There exists  $C > 0$  so that for any  $\mu \gg 1$*

$$|N(\mu, -\Delta_{A_j}^{\mathbf{X}_j}) - \frac{\omega_{n-1}}{(2\pi)^{n-1}} |\mathbf{X}_j| \mu^{(n-1)/2}| \leq C \mu^{(n-2)/2}. \quad (2.12)$$

**Lemma 2.3** *There exists  $C > 0$  such that for any  $\lambda \gg 1$*

$$|N(\lambda, -\Delta_A^{\mathbf{M}_j, D}) - \frac{\omega_n}{(2\pi)^n} |\mathbf{M}_j| \lambda^{n/2}| \leq \quad (2.13)$$

$$C \begin{cases} \lambda^{(n-1)/2} \ln(\lambda), & \text{if } 1/(n-1) \leq \delta_j \leq 1 \\ \lambda^{1/(2\delta_j)}, & \text{if } 1/n < \delta_j < 1/(n-1) \end{cases}.$$

**Proof of Lemma 2.3** By the formula (2.4),

$$N(\lambda, -\Delta_A^{\mathbf{M}_j, D}) = \sum_{\ell=0}^{+\infty} N(\lambda, L_{j,\ell}^D). \quad (2.14)$$

As  $N(\lambda, L_{j,\ell}^D) = 0$  when  $\ell \notin K_\lambda$ , ( $K_\lambda$  is defined in Lemma 2.1), the estimates (2.8), (2.12) and formula (2.14) prove that

$$|N(\lambda, -\Delta_A^{\mathbf{M}_j, D}) - \sum_{\ell=0}^{+\infty} \frac{1}{\pi} w_{j,\ell}(\lambda)| \leq C \lambda^{(n-1)/2} \ln(\lambda). \quad (2.15)$$

Let us denote

$$\Theta_j(\lambda) = \sum_{\ell=0}^{+\infty} \frac{1}{\pi} w_{j,\ell}(\lambda) \quad \text{and} \quad R_j(\mu) = \sum_{\ell=0}^{+\infty} [\mu - \mu_\ell(j)]_+^{1/2}. \quad (2.16)$$

$$\text{As } R_j(\mu) = \frac{1}{2} \int_0^{+\infty} [\mu - s]_+^{-1/2} N(s, -\Delta_{A_j}^{\mathbf{x}_j}) ds,$$

the Hörmander estimate (2.12) entails the following one.

There exists a constant  $C > 0$  such that, for any  $\mu \gg 1$ ,

$$|R_j(\mu) - \frac{\omega_{n-1}}{2(2\pi)^{n-1}} |\mathbf{X}_j| \int_0^{+\infty} [\mu - s]_+^{-1/2} s^{(n-1)/2} ds| \leq C \mu^{(n-1)/2}. \quad (2.17)$$

Writing in (2.9 )

$$V_{j,\ell}(t) = \mu_\ell(j) V_j(t) + W_j(t), \quad (2.18)$$

$$\text{we get that } \Theta_j(\lambda) = \frac{1}{\pi} \int_{\alpha_j}^{T_j(\lambda)} V_j^{1/2}(t) R_j\left(\frac{\lambda - W_j(t)}{V_j(t)}\right) dt.$$

So according to (2.17)

$$\begin{aligned} |\Theta_j(\lambda) - \frac{\omega_{n-1} \Gamma(\frac{1}{2}) \Gamma(\frac{n+1}{2})}{(2\pi)^n \Gamma(1 + \frac{n}{2})} |\mathbf{X}_j| \int_{\alpha_j}^{T_j(\lambda)} \frac{(\lambda - W_j(t))^{n/2}}{V_j^{(n-1)/2}(t)} dt| &\leq \\ &C \int_{\alpha_j}^{T_j(\lambda)} \frac{(\lambda - W_j(t))^{(n-1)/2}}{V_j^{(n-2)/2}(t)} dt. \end{aligned} \quad (2.19)$$



From the definitions (2.9) and (2.18) we get that

$$\left| \int_{\alpha_j}^{T_j(\lambda)} \frac{(\lambda - W_j(t))^{n/2}}{V_j^{(n-1)/2}(t)} dt - \lambda^{n/2} \frac{1}{(\delta_j n - 1) a_j^{2(\delta_j n - 1)}} \right| \leq C \lambda^{(n-1)/2}, \quad (2.20)$$

and

$$\int_{\alpha_j}^{T_j(\lambda)} \frac{(\lambda - W_j(t))^{(n-1)/2}}{V_j^{(n-2)/2}(t)} dt \leq \quad (2.21)$$

$$C \begin{cases} \lambda^{(n-1)/2} & \text{if } 1/(n-1) < \delta_j \leq 1 \\ \lambda^{(n-1)/2} \ln \lambda & \text{if } 1/(n-1) = \delta_j \\ \lambda^{1/(2\delta_j)} & \text{if } 1/n < \delta \leq 1/(n-1) \end{cases}.$$

As  $|\mathbf{M}_j| = \frac{|\mathbf{X}_j|}{(\delta_j n - 1) a_j^{2(\delta_j n - 1)}}$ , we get (2.13) from (2.15), (2.16) and (2.19)—  
(2.21)  $\square$

To achieve the proof of Theorem 1.2, we proceed as in [Mo-Tr].

We denote  $\mathbf{M}_0^0 = \mathbf{M} \setminus \left( \bigcup_{j=1}^J \overline{\mathbf{M}_j} \right)$ , then

$$\mathbf{M} = \overline{\mathbf{M}_0^0} \cup \left( \bigcup_{j=1}^J \overline{\mathbf{M}_j} \right). \quad (2.22)$$

Let us denote respectively by  $-\Delta_A^{\Omega, D}$  and by  $-\Delta_A^{\Omega, N}$  the Dirichlet operator and the Neumann-like operator on an open set  $\Omega$  of  $\mathbf{M}$  associated to  $-\Delta_A$ .  $-\Delta_A^{\Omega, N}$  is the Friedrichs extension defined by the associated quadratic form  $q_A^\Omega(u) = \int_\Omega |idu + Au|^2 d\mathbf{m}$ ,  $u \in C^\infty(\overline{\Omega}; \mathbb{C})$ ,  $u$  with compact support in  $\overline{\Omega}$ . ( $d\mathbf{m}$  is the  $n$ -form volume of  $\mathbf{M}$ ).

The minimax principle and (2.22) imply that

$$\begin{aligned} N(\lambda, -\Delta_A^{\mathbf{M}_0^0, D}) + \sum_{1 \leq j \leq J} N(\lambda, -\Delta_A^{\mathbf{M}_j, D}) &\leq N(\lambda, -\Delta_A) \\ &\leq N(\lambda, -\Delta_A^{\mathbf{M}_0^0, N}) + \sum_{1 \leq j \leq J} N(\lambda, -\Delta_A^{\mathbf{M}_j, N}) \end{aligned} \quad (2.23)$$

The Weyl formula with remainder, (see [Hor2] for Dirichlet boundary condition and [Sa-Va] p. 9 for Neumann-like boundary condition), gives that

$$N(\lambda, -\Delta_A^{\mathbf{M}_0^0, Z}) = \frac{\omega_n}{(2\pi)^n} |\mathbf{M}_0^0| \lambda^{n/2} + \mathbf{O}(\lambda^{(n-1)/2}) ; \quad (\text{for } Z = D \text{ and for } Z = N) . \quad (2.24)$$

For  $1 \leq j \leq J$ , the asymptotic formula for  $N(\lambda, -\Delta_A^{\mathbf{M}_j, N})$ ,

$$N(\lambda, -\Delta_A^{\mathbf{M}_j, N}) = \frac{\omega_n}{(2\pi)^n} |\mathbf{M}_j| \lambda^{n/2} + \mathbf{O}(r(\lambda)) , \quad (2.25)$$

is obtained as for the Dirichlet case (2.13) by noticing that  $N(\lambda, L_{j,\ell}^D) \leq N(\lambda, L_{j,\ell}^N) \leq N(\lambda, L_{j,\ell}^D) + 1$ , where  $L_{j,\ell}^D$  and  $L_{j,\ell}^N$  are Dirichlet and Neumann-like operators on a half-line  $I = ]\alpha_j, +\infty[$ , associated to the same differential Schrödinger operator  $L_{j,\ell}$  defined by (2.5) when  $\delta_j = 1$ , and by (2.7) otherwise. (The Neumann-like boundary condition is of the form  $\partial_t u(\alpha_j) + \beta_j u(\alpha_j) = 0$  because of the change of functions performed by  $U^*$ ).

We get (1.8) from (2.13) and (2.23)—(2.25)  $\square$

### 2.3 Proof of Theorem 1.3

**Lemma 2.4** *For any  $j \in \{1, \dots, J\}$ , there exists a one-form  $A_j$  satisfying (1.3) and the following property.*

*There exists  $\tau_0 = \tau_0(A_j) > 0$  and  $C = C(A_j) > 0$  such that for any  $\lambda \gg 1$ , if  $e(\tau, j) = \inf_{u \in C^\infty(\mathbf{X}_j), \|u\|_{L^2(\mathbf{X}_j)}=1} \|idu + \tau u A_j\|_{L^2(\mathbf{X}_j)}^2$  denotes the first eigenvalue of  $-\Delta_{\tau A_j}^{\mathbf{X}_j}$ , then*

$$e(\tau, j) \geq C\tau^2 ; \quad \forall \tau \in ]0, \tau_0] . \quad (2.26)$$

**Proof of Lemma 2.4.** When  $n = 2$ , we can take  $A_j = \omega_j d\mathbf{x}_j$ , ( $d\mathbf{x}_j$  is the  $(n-1)$ -form volume of  $\mathbf{X}_j$ ), for some constant  $\omega_j \in \mathbb{R} \setminus \frac{2\pi}{|\mathbf{X}_j|} \mathbb{Z}$ , then  $e(\tau, j) = \tau^2 \omega_j^2$  for small  $|\tau|$ .

When  $n \geq 3$ , we have  $e(0, j) = 0$ ,  $\partial_\tau e(0, j) = 0$  and

$$\partial_\tau^2 e(0, j) = \int_{\mathbf{X}_j} \left[ |A_j|^2 - (-\Delta_0^{\mathbf{X}_j})^{-1} (d^* A_j) \cdot (d^* A_j) \right] d\mathbf{x}_j .$$

( $d^*$  is the adjoint of  $d$  defined on functions, and  $(-\Delta_0^{\mathbf{X}_j})^{-1}$  is the inverse of the Laplace-Beltrami operator on functions, which is well-defined on the space  $\{f \in L^2(\mathbf{X}_j); \int_{\mathbf{X}_j} f d\mathbf{x}_j = 0\}$ ).

To the non-negative quadratic form  $A_j \rightarrow \partial_\tau^2 e(0, j)$ , we associate a self-adjoint operator  $P$  on  $T^*(\mathbf{X}_j)$ , which is a pseudodifferential operator of order 0 with principal symbol, the square matrix  $p_0(x, \xi) = (p_0^{ik}(x, \xi))_{1 \leq i, k \leq n-1}$  defined as follows. In local coordinates, if  $\mathbf{h}_j = G_{ik}(x) dx_i dx_k$ , then

$$p_0^{ik}(x, \xi) = G^{ik}(x) - \sum_{\ell, m} G^{im}(x) G^{\ell k}(x) \frac{\xi_m}{|\xi|} \frac{\xi_\ell}{|\xi|}; \quad (|\xi|^2 = \sum_{\ell, m} G^{m\ell}(x) \xi_m \xi_\ell).$$

As the non-negative symmetric matrix  $p_0(x, \xi)$  is not the zero matrix, there exists  $A_j$  such that  $P(A_j) \neq 0$  and by the positivity of  $P$ ,  $\partial_\tau^2 e(0, j) = \int_{\mathbf{X}_j} \langle P(A_j) | A_j \rangle d\mathbf{x}_j > 0$   $\square$

**Lemma 2.5** *For a one-form  $A$  satisfying (1.4), there exists a constant  $C_A > 0$  such that, if  $u$  is a function in  $L^2(\mathbf{M})$  such that  $du \in L^2(\mathbf{M})$  and*

$$\forall j = 1, \dots, J, \quad \int_{\mathbf{X}_j} u(x_j, y) d\mathbf{x}_j = 0, \quad \forall y \in ]a_j^2, +\infty[, \quad (2.27)$$

then  $\forall \tau \in ]0, 1]$ ,

$$\|idu + \tau u A\|_{L^2(\mathbf{M})}^2 \leq (1 + \tau C_A) \|idu\|_{L^2(\mathbf{M})}^2 + C_A \|u\|_{L^2(\mathbf{M})}^2. \quad (2.28)$$

**Proof of Lemma 2.5.** First we remark that the inequality

$$|idu + \tau u A|^2 \leq (1 + \rho) |du|^2 + (1 + \rho^{-1}) |\tau u A|^2 \quad (2.29)$$

is satisfied for any  $\rho > 0$ .

For  $\rho = \tau$  we get that there exists a constant  $C_A^0 > 0$ , depending only on  $A/\mathbf{M}_0$ , such that

$$\|idu + \tau u A\|_{L^2(\mathbf{M}_0)}^2 \leq (1 + \tau) \|idu\|_{L^2(\mathbf{M}_0)}^2 + \tau C_A^0 \|u\|_{L^2(\mathbf{M}_0)}^2. \quad (2.30)$$

We get also for  $\rho = \tau$  that for any  $j \in \{1, \dots, J\}$ ,

$$\int_{a_j^2}^{+\infty} \|idu + \tau u A\|_{L^2(\mathbf{X}_j)}^2 y^{(2-n)\delta_j} dy \leq \quad (2.31)$$

$$\int_{a_j^2}^{+\infty} \left( (1+\tau) \|idu\|_{L^2(\mathbf{X}_j)}^2 + \tau C_A^j \|u\|_{L^2(\mathbf{X}_j)}^2 \right) y^{(2-n)\delta_j} dy ,$$

for some constant  $C_A^j$  depending only on  $A/X_j$ .

But (2.27) implies that

$$\|u\|_{L^2(\mathbf{X}_j)}^2 \leq \frac{1}{\mu_1(j)} \|idu\|_{L^2(\mathbf{X}_j)}^2 , \quad (2.32)$$

with  $(\mu_\ell(j))_{\ell \in \mathbb{N}}$  the sequence of eigenvalues of Laplace-Beltrami operator on  $\mathbf{X}_j$ ,  $\mu_0(j) = 0 < \mu_1(j) \leq \mu_2(j) \leq \dots$ . So if (2.27) is satisfied then (2.31) and (2.32) imply that

$$\|idu + \tau u A\|_{L^2(\mathbf{M}_j)}^2 \leq (1 + \tau C_A^j) \|idu\|_{L^2(\mathbf{M}_j)}^2 , \quad (2.33)$$

for some constant  $C_A^j$  depending only on  $A/X_j$ .

The existence of a constant  $C_A > 0$  satisfying the inequality (2.28) follows from (2.30) and (2.33) for  $j = 1, \dots, J$   $\square$

**Lemma 2.6** *When  $A$  satisfies (1.3), (1.4) and Lemma 2.4 , then as  $\lambda \rightarrow +\infty$ , the following Weyl formula is satisfied.*

$$N(\lambda, -\Delta_{(\lambda^{-\rho} A)}) = |\mathbf{M}| \frac{\omega_n}{(2\pi)^n} \lambda^{n/2} + \mathbf{O}(\mathbf{r}_0(\lambda)) , \quad (2.34)$$

with

$$\rho = \begin{cases} 1/2, & \text{if } 2/n \leq \delta \leq 1 \\ (n\delta - 1)/2, & \text{if } 1/n < \delta < 2/n \end{cases} , \quad (2.35)$$

$\delta$  and  $\omega_d$  are as in Theorem 1.2, and the function  $r_0(\lambda)$  is the one defined by (1.11) .

**Proof of Lemma 2.6.** Since  $A$  satisfies Lemma 2.4, we have

$$C/\lambda^{2\rho} \leq \mu_0(j) \quad \text{and} \quad C \leq \mu_1(j) ,$$

where  $(\mu_\ell(j))_{\ell \in \mathbb{N}}$  denotes now the increasing sequence of eigenvalues of  $-\Delta_{\lambda^{-\rho} A_j}^{\mathbf{X}_j}$ . Hence we can mimick the proof of Theorem 1.2. More precisely Lemma 2.1 holds for any  $\ell \in K_\lambda$ ,  $\ell \neq 0$ , and to get the result it only remains to prove that we have, for  $L_{j,0}$  defined by (2.5) if  $\delta_j = 1$ , and by (2.7) otherwise,

$$N(\lambda, L_{j,0}^D) = \mathbf{O}(r_0(\lambda)) .$$

This can easily be done as follows.

When  $\delta_j = 1$ , ( $\rho = 1/2$ ), it is easy to see that

$$N(\lambda, L_{j,0}^D) \leq N(\lambda + C, L^{D,\lambda}) \leq C\lambda^{1/2} \ln(\lambda),$$

where  $L^{D,\lambda}$  is the Dirichlet operator on  $]0, +\infty[$  associated to  $\frac{C}{\lambda}e^{2t} - \partial_t^2$ .

When  $0 < \delta_j < 1$ , by scaling we have that

$$N(\lambda, L_{j,0}^D) \leq N((\lambda + C)^{1+2\rho(1-\delta_j)}, L^D) \leq C\lambda^{(1+2\rho(1-\delta_j))/(2\delta_j)},$$

where  $L^D$  is the Dirichlet operator on  $]0, +\infty[$  associated to  $\frac{1}{C^2}t^{\frac{2\delta_j}{1-\delta_j}} - \partial_t^2$ .

When  $2/n \leq \delta < 1$ , as  $2/n \leq \delta \leq \delta_j$ , then

$$\lambda^{(1+2\rho(1-\delta_j))/(2\delta_j)} = \lambda^{(2-\delta_j)/(2\delta_j)} \leq \lambda^{(2-\delta)/(2\delta)} \leq \lambda^{(n-1)/2} = \mathbf{O}(r_0(\lambda)).$$

When  $1/n < \delta < 2/n$ , as  $\delta \leq \delta_j$ , then

$$\lambda^{(1+2\rho(1-\delta_j))/(2\delta_j)} \leq \lambda^{(1+2\rho(1-\delta))/(2\delta)} = \lambda^{(n-(n\delta-1))/2} = \mathbf{O}(r_0(\lambda)) \quad \square$$

To achieve the proof of Theorem 1.3, we take a one-form  $A$  satisfying the assumptions of Lemma 2.6.

We remark that any eigenfunction  $u$  of the Laplace-Beltrami operator  $-\Delta$  on  $\mathbf{M}$  associated to an eigenvalue in  $] \inf \text{sp}_{\text{ess}}(-\Delta), +\infty[$ , satisfies (2.27). So if  $H_\lambda$  is the subspace of  $L^2(\mathbf{M})$  spanned by eigenfunctions of  $-\Delta$  associated to eigenvalues in  $]0, +\infty[$ , then, by (2.28) of Lemma 2.5 with  $\tau = 1/\lambda^\rho$ , with  $\rho$  defined by (2.35), we have

$$\forall u \in H_\lambda, \quad \|idu + \frac{1}{\lambda^\rho}uA\|_{L^2(\mathbf{M})}^2 \leq (1 + \frac{C_A}{\lambda^\rho})\|du\|_{L^2(\mathbf{M})}^2 + C_A\|u\|_{L^2(\mathbf{M})}^2 \quad (2.36)$$

So

$$\dim(H_\lambda) \leq N((1 + \frac{C_A}{\lambda^\rho})\lambda + C_A, -\Delta_{(\lambda^{-\rho}A)}). \quad (2.37)$$

The estimates (2.34) and (2.37) prove (1.10), by noticing that  $\lambda^{n/2}/\lambda^\rho = \mathbf{O}(r_0(\lambda)) \quad \square$

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